THE THERMAL BOUNDARY LAYER OF A NON-FOURIER POWER-LAW FLUID ON A PLATE WITH A VARIABLE SURFACE TEMPERATURE

S. L. Lur'e

Inzhenerno-Fizicheskii Zhurnal, Vol. 14, No. 3, pp. 505–510, 1968

UDC 536.24.532.526

The equation for the thermal boundary layer of a non-Fourier power-law fluid on a flat plate with an exponential distribution of surface temperature is reduced to an ordinary equation and solved by the method of finite differences. The effect of the exponent γ on the temperature profile and on the heat-transfer coefficient is determined. It is demonstrated that the asymptotic solutions of the equation for large σ are nearly exact.

With consideration of the non-Fourier heat-conduction law [1]

$$q = -H\left(\frac{\partial u_1}{\partial y_1}\right)^{n-1}\frac{\partial T}{\partial y_1} \tag{1}$$

the equation for the thermal boundary layer, with viscous dissipation neglected, is written in the form

$$u_1\frac{\partial T}{\partial x_1} + v_1\frac{\partial T}{\partial y_1} = \frac{H}{\rho c_p}\frac{\partial}{\partial y_1}\left[\left(\frac{\partial u_1}{\partial y_1}\right)^{n-1}\frac{\partial T}{\partial y_1}\right]. \tag{2}$$

We assume the following boundary conditions:

$$T = T_w(x_1) \text{ when } y_1 = 0;\quad T \to T_\infty \text{ as } y_1 \to \infty; \tag{3}$$

$$T_w(x_1) = T_\infty + Ax_1^\gamma. \tag{4}$$

Equation (2) is written as follows in dimensionless form:

$$\gamma\frac{u}{x}\theta + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = \frac{1}{\sigma}\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial\theta}{\partial y}\right]. \tag{5}$$

Here

$$u = \frac{u_1}{U},\quad v = \frac{v_1}{U}\,\mathrm{R}^{\frac{1}{1+n}},\quad x = \frac{x_1}{L},\quad y = \frac{y_1}{L}\,\mathrm{R}^{\frac{1}{1+n}},$$

$$\mathrm{R} = \frac{\rho U^{2-n}L^n}{K},$$

$$\theta = \frac{T - T_\infty}{T_w(x_1) - T_\infty},\quad \sigma = \frac{Kc_p}{H}. \tag{6}$$

For the boundary layer of a non-Newtonian power-law fluid on a flat plate, u and v are determined from the following formulas [2]:

$$u = \varphi'(\eta),\quad v = [n(1+n)x]^{\frac{1}{1+n}}[(1+n)x]^{-1}[\eta\varphi' - \varphi],$$

$$\eta = y[n(1+n)x]^{-\frac{1}{1+n}}, \tag{7}$$

where the prime denotes the derivative with respect to η, and the function φ satisfies the equation

$$\varphi''' + \varphi(\varphi'')^{2-n} = 0 \tag{8}$$

with the boundary conditions

$$\varphi = 0,\ \varphi' = 0 \text{ when } \eta = 0;\ \varphi' \to 1 \text{ as } \eta \to \infty. \tag{9}$$

With formulas (7), Eq. (5) is reduced to an ordinary differential equation

$$\theta'' + [(n-1)\varphi'''/\varphi'' + \sigma n(\varphi'')^{1-n}\varphi]\theta' - \sigma n(1+n)\gamma\varphi'(\varphi'')^{1-n}\theta = 0 \tag{10}$$

or with consideration of (8)

$$\theta'' + [(\sigma-1)n + 1](\varphi'')^{1-n}\varphi\theta' - \sigma n(1+n)\gamma\varphi'(\varphi'')^{1-n}\theta = 0. \tag{11}$$

Table 1

The Values of $-\theta'(0)$

n	σ	γ = −0.5	0	0.5	2.0	4.0
0.5	10	0.2431	0.5549	0.7155	0.9914	1.217
	100	0.5152	1.188	1.535	2.132	2.612
0.7	10	2.215	0.7555	0.9995	1.406	1.731
	100	0.4750	1.625	2.149	3.027	3.723
1.0	10	0.0000	1.030	1.411	2.016	2.490
	100	0.0000	2.223	3.041	4.345	5.364
1.5	10	−1.466	1.396	2.012	2.930	3.633
	100	−3.331	3.021	4.342	6.311	7.829
2.0	10	−29.31	1.662	2.503	3.692	4.591
	100	−567.0	3.602	5.404	7.958	9.890

The boundary condition (3) will be

$$\theta = 1 \text{ when } \eta = 0;\ \theta \to 0 \text{ as } \eta \to \infty. \quad (12)$$

Levy [3] dealt with an analogous problem for a Newtonian fluid and an arbitrary surface; the problem was formulated in [2] for a power-law fluid and for Eq. (1).

To solve the 2-nd order linear equation (11), we employ the method of finite differences used in [3], with slight modifications.

We will divide the region of integration into equal intervals of length h and, replacing the derivatives θ_i'' and θ_i' by the finite-difference ratios, instead of (11) we obtain

$$(\theta_{i+1} - 2\theta_i + \theta_{i-1})/h^2 + A_i(\theta_{i+1} - \theta_{i-1})/h - B_i\theta_i = 0, \quad (13)$$

where

$$A_i = [(\sigma - 1)n + 1](\varphi_i'')^{1-n}\varphi_i/2,$$
$$B_i = \sigma n(1+n)\gamma\varphi_i'(\varphi_i'')^{1-n}. \quad (14)$$

This yields the formula

$$\theta_{i+1} = a_i\theta_i - b_i\theta_{i-1} \quad (15)$$

when

$$a_i = (2 + h^2B_i)/(1 + hA_i),$$
$$b_i = (1 - hA_i)/(1 + hA_i). \quad (16)$$

Using formula (15) for i = 1, 2, 3, etc., with consideration of the boundary condition $\theta_0 = 1$, which follows from (12), we find that

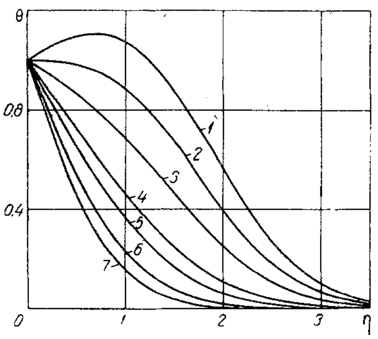

Fig. 1. Temperature profiles in the boundary layer at n = 0.5; $\sigma = 10$: 1) $\gamma = -0.75$; 2) $-2/3 = \gamma_0$; 3) -0.5; 4) 0; 5) 0.5; 6) 2.0; 7) 4.0.

$$\theta_{i+1} = c_i\theta_1 - d_i, \quad (17)$$

where

$$c_i = a_ic_{i-1} - b_ic_{i-2};\quad d_i = a_id_{i-1} - b_id_{i-2} \quad (18)$$

and

$$c_1 = a_1,\quad c_0 = 1,\quad c_{-1} = 0;$$
$$d_1 = b_1,\quad d_0 = 0,\quad d_{-1} = -1. \quad (19)$$

Applying the second boundary condition of (12) in the form of $\theta_{r+1} = 0$ to formula (17), with r corresponding to the external edge of the boundary layer, we have

$$\theta_1^{(r)} = d_r/c_r. \quad (20)$$

The value of i = r is obtained from the condition

$$|\theta_1^{(i)} - \theta_1^{(i-1)}|/h < \varepsilon, \quad (21)$$

where ε is a small number specified in advance. This

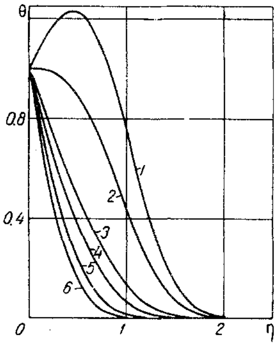

Fig. 2. Temperature profiles in boundary layer at n = 1.0; $\sigma = 10$: 1) $\gamma = -0.6$; 2) $-0.5 = \gamma_0$; 3) 0; 4) 0.5; 5) 2.0; 6) 4.0.

corresponds to the approach of the ratio d_i/c_i to some limit determining θ_1 in Eq. (20).

As a result, the solution of the problem is obtained with formulas (14) and (16)–(21). Initially the coefficients c_i and d_i are calculated according to formulas (18)–(19); the value of θ_1 from Eq. (20) is found, and then the profile θ from (17) is calculated. The solution was obtained on a Ural-2 computer; the solution of Eq. (8), required for the calculation of the coefficients in (14), was found by the Runge-Kutta method with consideration of the values of $\varphi''(0)$, which are known from [2].

The quantity $-\theta'(0)$, shown in Table 1, is determined from the formula

$$-\theta'(0) = (1 - \theta_1)/h. \quad (22)$$

Comparison of the values of $\theta'(0)$ with the corresponding quantities calculated by other methods for $\gamma = 0$ [2, 4] demonstrated that the described method, given proper selection of h and ε, yields no less than 4 exact significant figures. Good agreement with the results of [3] was not noted, although this can be explained by the fact that the accuracy of the difference equation corresponding to (13) is lower by an order of magnitude in [3] than in this paper [O(h) instead of $O(h^2)$], since a unilateral formula was used in [3] for the derivative θ_i' (in (13) a central formula was used) and the calculations were carried out for a constant and rather large pitch of h = 0.1. We note that in the calculation of $\theta'(0)$ in accordance with the unilateral formula (22) there is no loss of accuracy, since $\theta''(0) = 0$.

Examples of the calculated $\theta(\eta)$ profiles are shown in Figs. 1–3. We see that an increase in γ results in a refinement of thermal boundary layer. When $\gamma < 0$, the profiles exhibit a flexure point within the layer,

Table 2

Comparison of the Exact and Approximate Values of $-\theta'(0)$

β	n	σ = 10			σ = 100			γ
		exact	formula (33)	error, %	exact	formula (33)	error, %	
0.2735	0.5	0.6691	0.6664	−0.40	1.433	1.436	+0.21	0.3333
0.3567	0.7	0.9128	0.9121	−0.08	1.963	1.965	+0.10	0.2941
0.4696	1.0	1.246	1.248	+0.16	2.687	2.689	+0.07	0.25
0.6189	1.5	1.691	1.697	+0.35	3.653	3.656	+0.08	0.2
0.7265	2.0	2.014	2.023	+0.45	4.356	4.358	+0.05	0.1667

while for some $\gamma = \gamma_0$ the condition $\theta'(0) = 0$ is satisfied. When $\gamma < \gamma_0$, $\theta'(0) > 0$ and the temperature within the boundary layer exceeds the wall temperature ($\theta > 1$).

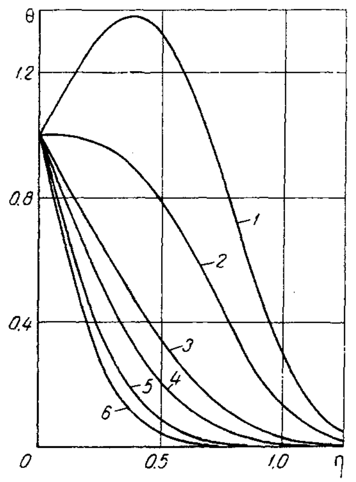

Fig. 3. Temperature profiles in the boundary layer at n = 1.5; σ = 10: 1) $\gamma = -0.5$; 2) $-0.4 = \gamma_0$; 3) 0; 4) 0.5; 5) 2.0; 6) 4.0.

Using formulas (1), (4), (6), and (7), we determine the local heat-transfer coefficient of the plate:

$$\mathrm{Nu}/R = -q_w K/AL^{\gamma} H\rho U = Ex_1^{\gamma} R_{x_1}^{-\frac{1}{1+n}}, \tag{23}$$

$$E = -[n(1+n)]^{-\frac{n}{1+n}} [\varphi''(0)]^{n-1}\theta'(0). \tag{24}$$

It follows from formula (23) that when $\gamma = n/(1+n)$, q_w is independent of x_1.

In view of the fact that the problem is solved for $\sigma \gg 1$, the thermal boundary layer is much thinner than the dynamic boundary layer and the following relationships of [4] are approximately valid:

$$\varphi'' = \text{const} = \beta, \quad \varphi''' = 0, \quad \varphi' = \beta\eta, \quad \varphi = \beta\eta^2/2. \tag{25}$$

Having substituted (25) into (10), we obtain the equation

$$\theta'' + a\eta^2\theta' + b\eta\theta = 0, \tag{26}$$

where

$$a = \sigma n\beta^{2-n}/2, \quad b = -\sigma n(1+n)\gamma\beta^{2-n}. \tag{27}$$

In the general case, Eq. (16), with substitution of the variables

$$\theta = \eta^{-1}\exp(-a\eta^3/6)W, \quad z = a\eta^3/3, \tag{28}$$

can be transformed into the Whittaker equation [5]

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{1/4 - m^2}{z^2}\right)W = 0 \tag{29}$$

when $k = (b - a)/3a$, $m = 1/6$, whose solution is expressed in the form of series in powers of z or in contour integrals with the parameter z. These solutions are complex for the calculations.

For certain relationships between b and a, Eq. (26) is easily solved under the conditions of (12). Thus, when $b = 2a$ we have the solution

$$\theta = \exp(-a\eta^3/3), \tag{30}$$

which corresponds to the case $\theta'(0) = 0$ and therefore defines the quantity γ_0. It is precisely when $b = 2a$ that it follows from (27) that

$$\gamma_0 = -1/(1+n). \tag{31}$$

Calculations have shown that formula (31) for the values of σ and n under consideration within the limitations of 4 significant figures is exact, while the profiles of (30) virtually do not differ from the results of the numerical solution.

Another simple solution is derived for $b = -a$, i.e., when $\gamma = 1/2(1+n)$, and this solution is in the form

$$\theta = \exp(-a\eta^3/3) - a\eta\int_{\eta}^{\infty}\eta\exp(-a\eta^3/3)\,d\eta, \tag{32}$$

whence

$$-\theta'(0) = (a/3)^{1/3}\,\Gamma(2/3) \approx 0.7452\,(\sigma n\beta^{2-n})^{1/3}. \tag{33}$$

The accuracy of formula (33) is sufficiently high, as can be seen from Table 2.

The solution for the case $b = 0$ ($\gamma = 0$) was given in [4] and it was also close to an exact solution.

The asymptotic equation (26) thus yields a solution that is little different from the solution of Eq. (10).

NOTATION

q is the heat flux in the boundary layer; q_w is the same, at the wall; H is the thermal conductivity characteristic; K and n are the rheological characteristics of the fluid; x_1 is the longitudinal coordinate; y_1 is the transverse coordinate; u_1 and v_1 are the projections of the velocity vector onto the x_1- and y_1-axes, respectively; U is the velocity of external flow; L is the characteristic length; R is the Reynolds number; ρ is the fluid density; c_p is the specific heat capacity; A and γ are the constants (formula (4)); σ is the Prandtl number; Nu is the Nusselt number.

REFERENCES

1. K. I. Strakhovich and S. L. Lur'e, IFZh [Journal of Engineering Physics], **12**, no. 3, 1967.

2. Z. P. Shul'man and B. M. Berkovskii, The Boundary Layer of Non-Newtonian Fluids [in Russian], Izd. Nauka i tekhnika, Minsk, 1966.

3. S. Levy, Journ. Aeron. Sci., **19**, no. 5, 341, 1952.

4. S. L. Lur'e, IFZh [Journal of Engineering Physics], **14**, no. 1, 1968.

5. E. T. Whittaker and J. Watson, Modern Analysis, Part 2 [Russian translation], Fizmatgiz, Moscow, 1963.

27 March 1967

Kalinin Polytechnic Institute, Leningrad